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CLASSES OF MEROMORPHIC α-CONVEX FUNCTIONS

Rosihan M. Ali and V. Ravichandran

Abstract. For a fixed analytic univalent function ϕ, the class of meromorphic univalent α-convex functions with respect to ϕ is introduced. A representation theorem for functions in the class, as well as a necessary and sufficient condition for functions to belong to the class are obtained. Also we obtain a sharp growth theorem and estimate on a certain coefficient functional for meromorphic starlike functions with respect to ϕ. Differential subordination and superordination conditions are also obtained for the subclass of meromorphic starlike functions with respect to ϕ.

1. INTRODUCTION

Let Σ denote the class of meromorphic univalent functions f defined on the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ having the form $f(z) = 1/z + \sum_{k=0}^{\infty} a_k z^k$. A function $f \in \Sigma$ is said to be *meromorphic starlike of order* α $(0 \le \alpha < 1)$ if $-\Re[zf'(z)/f(z)] > \alpha$ for all $z \in \Delta := \{z \in \mathbb{C} : |z| < 1\}$. We denote by $\Sigma^*(\alpha)$ the class of all such meromorphic starlike functions of order α in Δ^*.

Several authors [2, 3, 7, 10, 11, 14, 16, 17] have studied various subclasses of $\Sigma^*(\alpha)$, as well as subclasses of *meromorphic convex functions of order* α. The latter class is characterized by the property $-\Re[1 + zf''(z)/f'(z)] > \alpha$. We shall unify these functions in Definition 1.1.

First we recall the definition of subordination. For two functions f and g analytic in Δ, we say that the function $f(z)$ is *subordinate* to $g(z)$ in Δ, and write $f \prec g$ or $f(z) \prec g(z)$ $(z \in \Delta)$, if there exists a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \Delta)$, such that $f(z) = g\big(w(z)\big)$ $(z \in \Delta)$. In particular, if the function g is *univalent* in Δ, the above subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

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Definition 1.1. *Let $\phi(z)$ be an analytic univalent function in Δ with $\phi(0) = 1$. Let $\Sigma^*_\alpha(\phi)$ be the class of functions $f \in \Sigma$ satisfying $f(z)f'(z) \neq 0$ and*

$$-\left[(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \prec \phi(z) \quad (z \in \Delta). \tag{1.1}$$

The function $f \in \Sigma^*_\alpha(\phi)$ is called a meromorphic α-convex function with respect to ϕ. (Here $\prec$ denotes subordination between analytic functions.) We shall write $\Sigma^*_0(\phi)$ by $\Sigma^*(\phi)$.

With

$$\phi(z) = \frac{1 + (1-2\alpha)z}{1-z} \quad (0 \leq \alpha < 1),$$

it is obvious that $\Sigma^*_0(\phi)$ is the class of meromorphic starlike functions of order α, while $\Sigma^*_1(\phi)$ is the class of meromorphic convex functions of order α. The class $\Sigma^*(\phi)$ reduces to the class $\Sigma(\alpha, \beta, \gamma)$ introduced by Kulkarni and Joshi [5] when

$$\phi(z) = \frac{1 + \beta(1-2\alpha\gamma)z}{1 + \beta(1-2\gamma)z} \quad (0 \leq \alpha < 1;\ 0 < \beta \leq 1;\ 1/2 \leq \gamma \leq 1). \tag{1.2}$$

Karunakaran [4] have considered a special case of the class $\Sigma^*(\phi)$ consisting of functions $f \in \Sigma$ for which

$$-\frac{zf'(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (0 \leq B < 1; -B < A < B),$$

where $w(z)$ is an analytic function in Δ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \Delta)$. He denoted this class by $K_1(A, B)$.

In this paper, a representation theorem as well as a necessary and sufficient condition for functions to belong to $\Sigma^*_\alpha(\phi)$ is obtained. Also we obtain a sharp growth theorem and estimate for the coefficient functional $|a_1 - \mu a_0^2|$ for functions in $\Sigma^*(\phi)$. Finally we investigate the subclass $\Sigma^*(\phi)$ from the perspective of first-order differential subordination and superordination [8, 9].

2. A Representation Theorem

We first prove a representation formula for functions in the class $\Sigma^*_\alpha(\phi)$.

Theorem 2.1. *A function $f(z) \in \Sigma^*_\alpha(\phi)$ if and only if*

$$[zf(z)]^{1-\alpha}[-z^2 f'(z)]^\alpha = \exp\left(\int_0^z \frac{1 - \phi(w(\eta))}{\eta} d\eta\right),$$

where $w(z)$ is analytic in Δ satisfying $w(0) = 0$ and $|w(z)| \leq 1$.

Proof. Let $f(z) \in \Sigma_\alpha^*(\phi)$. Then (1.1) holds and therefore there is a function $w(z)$ analytic in Δ with $w(0) = 0$ and $|w(z)| \leq 1$ such that

$$-\left[(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] = \phi(w(z)), \quad (z \in \Delta).$$

Rewriting the above equation in the form

$$\left[(1-\alpha)\left(\frac{1}{z} + \frac{f'(z)}{f(z)}\right) + \alpha\left(\frac{2}{z} + \frac{f''(z)}{f'(z)}\right)\right] = \frac{1-\phi(w(z))}{z}, \quad (z \in \Delta)$$

and integrating from 0 to z, we obtain the desired expression upon exponentiation. The converse follows directly by differentiation. ∎

Example 2.1. For the function $\phi(z)$ given by (1.2) and with $\alpha = 0$, we obtain [5, Theorem 1, p. 198]: Let $f \in \Sigma$ and $0 \leq \alpha < 1$, $0 < \beta \leq 1$ and $1/2 \leq \gamma \leq 1$. Then $f \in \Sigma(\alpha, \beta, \gamma)$ if and only if

$$zf(z) = \exp\left(-\int_0^z \frac{2\beta\gamma(1-\alpha)w(\eta)}{[1+\beta(1-2\gamma)w(\eta)]\eta} d\eta\right)$$

where $w(z)$ is analytic in Δ satisfying $w(0) = 0$ and $|w(z)| \leq 1$.

3. A Necessary and Sufficient Condition

We need the following subordination result.

Lemma 3.1. [13]. *Let ϕ be a convex univalent function defined on Δ and $\phi(0)=1$. Define $F(z)$ by*

$$F(z) = z \exp\left(\int_0^z \frac{\phi(\eta)-1}{\eta} d\eta\right).$$

Let $q(z)$ be analytic in Δ and $q(0) = 1$. Then

$$1 + \frac{zq'(z)}{q(z)} \prec \phi(z) \tag{3.1}$$

if and only if for all $|s| \leq 1$ and $|t| \leq 1$,

$$\frac{q(tz)}{q(sz)} \prec \frac{sF(tz)}{tF(sz)}. \tag{3.2}$$

Using Lemma 3.1, we obtain the following necessary and sufficient conditions for functions to belong to $\Sigma_\alpha^*(\phi)$.

Theorem 3.1. *Let $\phi(z)$ and $F(z)$ be as in Lemma 3.1. A function f belongs to $\Sigma^*_\alpha(\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$,*

$$\left(\frac{sf(sz)}{tf(tz)}\right)^{1-\alpha}\left(\frac{s^2f'(sz)}{t^2f'(tz)}\right)^{\alpha} \prec \frac{sF(tz)}{tF(sz)}.$$

Proof. Define the function $q(z)$ by

$$\frac{1}{q(z)} := (zf(z))^{1-\alpha}\left(-z^2f'(z)\right)^{\alpha}.$$

Then a computation shows that

$$1+\frac{zq'(z)}{q(z)} = -\left[(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]$$

and the result now follows from Lemma 3.1. ∎

Example 3.2. *Let $\Sigma^*_\alpha[A,B]$ be the class of all meromorphic α-convex functions $f \in \Sigma$ satisfying*

$$-\left[(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] \prec \frac{1+Az}{1+Bz} \quad (-1\leq B<A\leq 1;\, z\in\Delta).$$

The function $f \in \Sigma^*_\alpha[A,B]$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$,

$$\left(\frac{sf(sz)}{tf(tz)}\right)^{1-\alpha}\left(\frac{s^2f'(sz)}{t^2f'(tz)}\right)^{\alpha} \prec \begin{cases} \left(\dfrac{1+Btz}{1+Bsz}\right)^{(A-B)/B} & \text{if } B \neq 0 \\ e^{A(t-s)z} & \text{if } B = 0 \end{cases}.$$

4. Growth Theorem for Functions in $\Sigma^*(\phi)$

We need the following Lemma in the proof of Theorem 4.1.

Lemma 4.1. [8, Corollary 3.4h.1, p.135]. *Let $q(z)$ be univalent in Δ and let $\psi(z)$ be analytic in a domain containing $q(\Delta)$. If $zq'(z)/\psi(q(z))$ is starlike, and*

$$zp'(z)\psi(p(z)) \prec zq'(z)\psi(q(z)),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Theorem 4.1 below is a special case of Theorem 3.1 if ϕ is a convex univalent function. However we prove Theorem 4.1 without the convexity assumption.

Theorem 4.1. *Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ and maps the unit disk Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let the functions $h_{\phi n}$ $(n = 2, 3, \ldots)$ be defined by*

$$\frac{zh'_\phi(z)}{h_\phi(z)} = \phi(z) \quad (h_\phi(0) = 0 = h'_\phi(0) - 1).$$

If $f(z) \in \Sigma^*(\phi)$, then

$$zf(z) \prec \frac{z}{h_\phi(z)}.$$

Proof. Define the function $p(z)$ by

$$p(z) := zf(z) \quad (z \in \Delta).$$

Then a computation shows that

$$-\frac{zf'(z)}{f(z)} = 1 - \frac{zp'(z)}{p(z)}.$$

If $f(z) \in \Sigma^*(\phi)$, then

$$\frac{zp'(z)}{p(z)} \prec 1 - \phi(z).$$

Since $\phi(z)$ is starlike in Δ, by an application of Lemma 4.1, we obtain $p(z) \prec q(z)$ where $q(z)$ is given by

$$\frac{zq'(z)}{q(z)} = 1 - \frac{zh'_\phi(z)}{h_\phi(z)}$$

or $q(z) = z/h_\phi(z)$. ∎

As a consequence of Theorem 4.1, we immediately obtain

Theorem 4.2. (Growth Theorem). *Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ and maps the unit disk Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. If $f(z) \in \Sigma^*(\phi)$, then*

$$[h_\phi(r)]^{-1} \leq |f(z)| \leq [-h_\phi(-r)]^{-1} \quad (|z| = r < 1).$$

For the choice $p(z) = (1 - Az)/(1 - Bz)$, $0 \leq B \leq 1$; $-B < A < B$, we obtain the following result of Karanukaran:

Corollary 4.1. [4]. *If $f \in K_1(A,B)$, then*

$$r^{-1}(1-Br)^{(B-A)/B} \leq |f(z)| \leq r^{-1}(1+Br)^{(B-A)/B}$$

5. Coefficient Problem for the Class $\Sigma^*(\phi)$

Now we consider coefficient problems for the class $\Sigma^*(\phi)$.

Theorem 5.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z) = 1/z + \sum_{k=0}^{\infty} a_k z^k$ belongs to $\Sigma^*(\phi)$, then*

$$|a_1 - \mu a_0^2| \leq \begin{cases} \frac{1}{2}(B_1^2 - 2\mu B_1^2 - B_2) & \text{if} \quad 2\mu B_1^2 \leq B_1^2 - B_1 - B_2 \\ \frac{1}{2}B_1 & \text{if} \quad B_1^2 - B_1 - B_2 \leq 2\mu B_1^2 \leq B_1^2 + B_1 - B_2. \\ \frac{1}{2}(-B_1^2 + 2\mu B_1^2 + B_2) & \text{if} \quad B_1^2 + B_1 - B_2 \leq 2\mu B_1^2 \end{cases}$$

The result is sharp.

Proof. Our proof of Theorem 5.1 is essentially similar to the proof of Theorem 3 of Ma and Minda[6]. If $f(z) \in \Sigma^*(\phi)$, then there is a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that

$$-\frac{zf'(z)}{f(z)} = \phi(w(z)). \tag{5.1}$$

Define the function $p_1(z)$ by

$$p_1(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots . \tag{5.2}$$

Since $w(z)$ is a Schwarz function, we see that $\Re p_1(z) > 0$ and $p_1(0) = 1$. Define the function $p(z)$ by

$$p(z) := -\frac{zf'(z)}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots . \tag{5.3}$$

In view of the equations (5.1), (5.2), (5.3), we have

$$p(z) = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right). \tag{5.4}$$

and from this equation (5.4), we obtain

$$b_1 = \frac{1}{2} B_1 c_1$$

and

$$b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2c_1^2.$$

From the equation (5.3), we see that

$$b_1 + a_0 = 0 \tag{5.5}$$

$$b_2 + b_1a_0 + 2a_1 = 0 \tag{5.6}$$

or equivalently

$$a_0 = -b_1 = -\frac{B_1c_1}{2} \tag{5.7}$$

and

$$\begin{aligned} a_1 &= \frac{1}{2}(b_1^2 - b_2) \\ &= \frac{1}{8}\left\{B_1^2c_1^2 - 2B_1c_2 + B_1c_1^2 - B_2c_1^2\right\}. \end{aligned}$$

Therefore,

$$a_1 - \mu a_0^2 = -\frac{B_1}{4}\left\{c_2 - vc_1^2\right\} \tag{5.8}$$

where

$$v := \frac{1}{2}\left[1 + B_1 - \frac{B_2}{B_1} - 2\mu B_1\right].$$

Our result now follows by an application of Lemma 5.2 below. The sharpness is also an immediate consequence of Lemma 5.2. ∎

Lemma 5.2. [6]. *If $p_1(z) = 1 + c_1z + c_2z^2 + \cdots$ is a function with positive real part in Δ, then*

$$|c_2 - vc_1^2| \le \begin{cases} -4v + 2 & \text{if } v \le 0 \\ 2 & \text{if } 0 \le v \le 1 \\ 4v - 2 & \text{if } v \ge 1 \end{cases}$$

When $v < 0$ or $v > 1$, equality holds if and only if $p_1(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < v < 1$, then equality holds if and only if $p_1(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $v = 0$, equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \le \lambda \le 1)$$

or one of its rotations. If $v = 1$, equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.

When μ is complex, we have the following:

Theorem 5.2. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *If* $f(z) = 1/z + \sum_{k=0}^{\infty} a_k z^k$ *belongs to* $\Sigma^*(\phi)$, *then for* μ *a complex number,*

$$|a_1 - \mu a_0^2| \leq \frac{B_1}{2} \max\{1, |B_1 - 2\mu B_1 - \frac{B_2}{B_1}|\}.$$

The result is sharp.

Theorem 5.2 follows from the following result. For a function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ with positive real part, we have

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

6. Differential Subordination and Superordination for $\Sigma^*(\phi)$

In this section, we discuss differential implications for the subclass $\Sigma^*(\phi)$. We shall require the following definition and lemmas:

Definition 6.1. [9, Definition 2, p. 817]. *Denote by* $\mathcal{Q}$, *the set of all functions* $f(z)$ *that are analytic and injective on* $\overline{\Delta} - E(f)$, *where*

$$E(f) = \{\zeta \in \partial\Delta : \lim_{z \to \zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ *for* $\zeta \in \partial\Delta - E(f)$.

Lemma 6.1. (cf. Miller and Mocanu [8, Theorem 3.4h, p. 132]). *Let* $q(z)$ *be univalent in the unit disk* Δ *and* ϑ *and* φ *be analytic in a domain* D *containing* $q(\Delta)$ *with* $\varphi(w) \neq 0$ *when* $w \in q(\Delta)$. *Set* $Q(z) := zq'(z)\varphi(q(z))$ *and* $h(z) := \vartheta(q(z)) + Q(z)$. *Suppose that either* $h(z)$ *is convex, or* $Q(z)$ *is starlike univalent in* Δ. *In addition, assume that* $\Re[zh'(z)/Q(z)] > 0$ *for* $z \in \Delta$. *If* $p(z)$ *is analytic in* Δ *with* $p(0) = q(0)$, $p(\Delta) \subseteq D$ *and*

$$\vartheta(p(z)) + zp'(z)\varphi(p(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)), \tag{6.1}$$

then $p(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Lemma 6.2. [1]. *Let $q(z)$ be univalent in the unit disk Δ and ϑ and φ be analytic in a domain D containing $q(\Delta)$. Suppose that $\Re\left[\vartheta'(q(z))/\varphi(q(z))\right] > 0$ for $z \in \Delta$ and $zq'(z)\varphi(q(z))$ is starlike univalent in Δ. If $p(z) \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$, with $p(\Delta) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in Δ, then*

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)),$$

implies $q(z) \prec p(z)$ and $q(z)$ is the best subordinant. (Here $\mathcal{H}[a, n]$ denotes the class of all analytic functions $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ $(z \in \Delta)$.)

First we prove a differential subordination result for the class $\Sigma^*(\phi)$.

Theorem 6.1. *Let α be a nonzero complex number. Let $q(z)$ be univalent in Δ, $q(0) = 1$. Assume that $q(z)$ or $(\alpha - 1)q(z) + \alpha q^2(z) - \alpha zq'(z)$ is convex univalent and*

$$\Re\left\{\frac{1-\alpha}{\alpha} - 2q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0. \tag{6.2}$$

If $f \in \Sigma$ satisfies

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec (\alpha - 1)q(z) + \alpha q^2(z) - \alpha zq'(z),$$

then $-\frac{zf'(z)}{f(z)} \prec q(z)$ and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := -\frac{zf'(z)}{f(z)}. \tag{6.3}$$

Then a computation shows that

$$p(z) - \frac{zp'(z)}{p(z)} = -\left(1 + \frac{zf''(z)}{f'(z)}\right). \tag{6.4}$$

Using (6.4) and (6.3), we have

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} = (\alpha - 1)p(z) + \alpha p^2(z) - \alpha zp'(z).$$

Define the function ϑ and φ by

$$\vartheta(w) = (\alpha - 1)w + \alpha w^2 \quad \text{and} \quad \varphi(w) = -\alpha.$$

Then the functions ϑ and φ are analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$ in $\mathbb{C}$. Also the function $Q(z) := zq'(z)\varphi(q(z)) = -\alpha zq'(z)$ is starlike in Δ. Using (6.2), we see that the function $h(z) := \vartheta(q(z)) + Q(z) = (\alpha - 1)q(z) + \alpha q(z)^2 + Q(z)$ satisfies $\Re[zh'(z)/Q(z)] > 0$. The result now follows by an application of Lemma 6.1. ∎

Theorem 6.2. *Let* $q(z) \neq 0$ *be univalent in* Δ *and* $q(0) = 1$. *Let* $zq'(z)/q(z)^2$ *be starlike in* Δ. *If* $f \in \Sigma$ *and*

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 - \frac{zq'(z)}{q(z)^2},$$

then $-zf'(z)/f(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Proof. Let $p(z)$ be defined by (6.3). From (6.4) and (6.3), we get

$$1 - \frac{zp'(z)}{p(z)^2} = \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)}$$

and the result follows by an application of Lemma 6.1. ∎

The corresponding superordination results are obtained from Lemma 6.2 in a similar manner to Theorems 6.1 and 6.2. The proofs are omitted.

Theorem 6.3. *Let* α *be a nonzero complex number,* $q(z)$ *be convex univalent in* Δ. *Assume that* $\Re\left\{\frac{\alpha-1}{\alpha} + 2q(z)\right\} < 0$. *If* $f \in \Sigma$, $-zf'(z)/f(z) \in \mathcal{H}[1,1] \cap \mathcal{Q}$ *and* $\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)}$ *is univalent in* Δ *and*

$$(\alpha - 1)q(z) + \alpha q^2(z) - \alpha z q'(z) \prec \frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)},$$

then $q(z) \prec -\frac{zf'(z)}{f(z)}$ *and* $q(z)$ *is the best subordinant.*

Theorem 6.4. *Let* $q(z) \neq 0$ *be univalent,* $q(0) = 1$ *and* $zq'(z)/q(z)^2$ *be starlike in* Δ. *If* $f \in \Sigma$, $-zf'(z)/f(z) \in \mathcal{H}[1,1] \cap \mathcal{Q}$ *and* $\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}$ *is univalent in* Δ

$$1 - \frac{zq'(z)}{q(z)^2} \prec \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)},$$

then $q(z) \prec -zf'(z)/f(z)$ *and* $q(z)$ *is the best subordinant.*

References

1. T. Bulboaca, Classes of first-order differential superordinations, *Demonstratio Math.*, **35(2)** (2002), 287-292.

2. J. Clunie, On meromorphic schlicht functions, *J. London Math. Soc.*, **34** (1959), 215-216.

3. N. E. Cho and S. Owa, Sufficient conditions for meromorphic starlikeness and close-to-convexity of order α, *Int. J. Math. Math. Sci.*, **26(5)** (2001), 317-319.

4. V. Karunakaran, On a class of meromorphic starlike functions in the unit disc, *Math. Chronicle*, **4(2-3)** (1976), 112-121.

5. S. R. Kulkarni and Sou. S. S. Joshi, On a subclass of meromorphic univalent functions with positive coefficients, *J. Indian Acad. Math.*, **24(1)** (2002), 197-205.

6. W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in: *Proceedings of the Conference on Complex Analysis* (*Tianjin, 1992*), 157-169, Internat. Press, Cambridge, MA.

7. J. Miller, Convex meromorphic mappings and related functions, *Proc. Amer. Math. Soc.*, **25** (1970), 220-228.

8. S. S. Miller and P. T. Mocanu, *Differential Subordinations*: *Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics (No. 225), Marcel Dekker, New York, 2000.

9. S. S. Miller and P. T. Mocanu, Subordinants of differential superordinations, *Complex Var. Theory Appl.* **48(10)** (2003), 815-826.

10. M. Nunokawa and O. P. Ahuja, On meromorphic starlike and convex functions, *Indian J. Pure Appl. Math.*, **32(7)** (2001), 1027-1032.

11. Ch. Pommerenke, On meromorphic starlike functions, *Pacific J. Math.*, **13** (1963), 221-235.

12. V. Ravichandran, S. Sivaprasad Kumar and M. Darus, On a subordination theorem for a class of meromorphic functions, *J. Inequal. Pure Appl. Math.*, **5(1)** (2004), Article 8, 4 pp. (electronic).

13. V. Ravichandran, M. Bolcal, Y. Polotoglu and A. Sen, Certain subclasses of starlike and convex functions of complex order, *Hacet. J. Math. Stat.*, **34** (2005), 9-15.

14. W. C. Royster, Meromorphic starlike multivalent functions, *Trans. Amer. Math. Soc.*, **107** (1963), 300-308.

15. S. Ruscheweyh, *Convolutions in Geometric Function Theory*, Presses Univ. Montréal, Montreal, Que., 1982.

16. H. M. Srivastava and S. Owa (eds.), *Current Topics in Analytic Function Theory*, World Sci. Publishing, River Edge, NJ, 1992.

17. B. A. Uralegaddi and A. R. Desai, Integrals of meromorphic starlike functions with positive and fixed second coefficients, *J. Indian Acad. Math.*, **24(1)** (2002), 27-36.

Rosihan M. Ali
School of Mathematical Sciences,
Universiti Sains Malaysia,
11800 USM Penang,
Malaysia
E-mail: rosihan@cs.usm.my

V. Ravichandran
Department of Mathematics,
University of Delhi,
Delhi 110 007,
India
E-mail: vravi@maths.du.ac.in
vravi68@gmail.com